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LETTER TO THE EDITOR

On the critical behaviour of triplet order parameters and susceptibilities in Ising models

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Abstract. Using exact inequalities, we prove, for d-dimensional Ising models with ferromagnetic pair interactions, that: (i) any triplet order parameter $\langle\sigma_1\sigma_2\sigma_3\rangle$ vanishes at the critical temperature with the same exponent as the conventional order parameter $\langle\sigma_1\rangle$; and (ii) the associated susceptibilities diverge as $T \to T_c$ from above with the same exponents. The first result confirms a recent conjecture of Wood and Griffiths based on series expansions.

We consider an Ising model on a d-dimensional lattice with a reduced Hamiltonian of the form (Wood and Griffiths 1973)

$$-\beta\mathcal{H} = K_1 \sum \sigma_1 + K_2 \sum \sigma_i\sigma_j + K_3 \sum \sigma_i\sigma_j\sigma_k, \tag{1}$$

where the first sum is over all N sites of the lattice, the second runs over all the nearest-neighbour bonds and the third over all the elementary triangles of the lattice. A specific example of such a system is the Ising model on the triangular lattice ($d = 2$) with pair and triplet interactions (Wood and Griffiths 1972, Sykes and Watts 1975). For this case, Baxter (1975) has evaluated the triplet order parameter.

$$M_3 = \langle\sigma_1\sigma_2\sigma_3\rangle = \tfrac{1}{3} \lim_{K_3\to 0+} [\partial(-\beta f)/\partial K_3]_{K_1=0}, \tag{2}$$

where f is the free energy per spin and, in the following, all ensemble averages $\langle\cdot\cdot\cdot\rangle$ will be understood to be calculated using (1) with $K_3 = 0$. At the critical temperature $K_2 = K_c$, M_3 was found to vanish with an exponent β_3 equal to that of the conventional spontaneous magnetization, i.e. $\beta_3 = \beta = 1/8$. More recently, Wood and Griffiths (1976) have derived series expansions for M_3 on the face-centred and body-centred cubic lattices ($d = 3$). Analysis of these series led Wood and Griffiths to conjecture that $\beta_3 = \beta$ for all d.

In this letter we confirm this conjecture using a simple argument based on exact inequalities which are valid for Ising models. In addition, we investigate the behaviour of the associated 'triplet susceptibility'

$$\chi_3 = [\partial^2(-\beta f)/\partial^2 K_3]_{K_1=K_3=0} = \sum_\alpha (\langle s_0^3 s_\alpha^3\rangle - \langle s_0^3\rangle\langle s_\alpha^3\rangle), \tag{3}$$

where $s_\alpha^3 = \sigma_\alpha^1\sigma_\alpha^2\sigma_\alpha^3$ is the product of the three spins at the vertices of triangle α, centred at position $\boldsymbol{r}_\alpha$.

We begin by proving the result $\beta_3=\beta$ for all d. A lower bound on M_3 follows immediately from the GKS inequality (Griffiths 1967, Kelly and Sherman 1968), namely

$$\langle\sigma_1\sigma_2\sigma_3\rangle\geq\langle\sigma_1\sigma_2\rangle\langle\sigma_3\rangle, \qquad (K_1\geq 0). \tag{4}$$

To obtain an upper bound, we use the GHS inequality (Griffiths *et al* 1970), which states that

$$u_3(i,j,k)\equiv\langle\sigma_i\sigma_j\sigma_k\rangle-\langle\sigma_i\sigma_j\rangle\langle\sigma_k\rangle-\langle\sigma_i\sigma_k\rangle\langle\sigma_j\rangle-\langle\sigma_j\sigma_k\rangle\langle\sigma_i\rangle+2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle\leq 0, \quad (K_1\geq 0). \tag{5}$$

Re-arranging this result and combining with (4) yields in the limit $K_1\to 0$,

$$\Gamma_0 M_0\leq M_3\leq M_0(3\Gamma_0-2M_0^2), \tag{6}$$

where $M_0=\lim_{K_1\to 0+}\langle\sigma_i\rangle$ is the spontaneous magnetization and $\Gamma_0=\lim_{K_1\to 0+}\langle\sigma_i\sigma_j\rangle$ is the nearest-neighbour correlation. Since Γ_0 is finite at the critical point (Γ_0 is essentially the internal energy per spin), (6) immediately implies the required result.

In passing, we note that the inequality $\beta_3=\beta$ is not restricted to the particular triplet order parameter defined by (2), but applies to any triplet correlation $\langle\sigma_i\sigma_j\sigma_k\rangle$. In particular, the bounds (6) remain valid with Γ_0 replaced by some other appropriate pair correlation, which however is finite at T_c.

Turning now to the critical behaviour of χ_3, we first use a result of Lebowitz (1972) to construct an upper bound proportional to the conventional magnetic susceptibility χ. Let A and B be two sets of lattice sites. For such sets we define

$$S_A=\sum_{i\in A}\rho_i, \qquad \rho_A=\prod_{i\in A}\rho_i, \tag{7}$$

where

$$\rho_i=\tfrac{1}{2}(\sigma_i+1)=0,1. \tag{8}$$

Then Lebowitz (1972) has proved that

$$0\leq\langle\rho_A\rho_B\rangle-\langle\rho_A\rangle\langle\rho_B\rangle\leq\langle S_A\rho_B\rangle-\langle S_A\rangle\langle\rho_B\rangle\leq\langle S_AS_B\rangle-\langle S_A\rangle\langle S_B\rangle=\sum_{i\in A}\sum_{j\in B}u_2(i,j), \tag{9}$$

where

$$u_2(i,j)=\langle\rho_i\rho_j\rangle-\langle\rho_i\rangle\langle\rho_j\rangle=\tfrac{1}{4}(\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle). \tag{10}$$

If we specialize to the case where the sets A and B are the sites at the vertices of triangles whose centres are at $\boldsymbol{r}_\alpha$ and $\boldsymbol{r}_\beta$ respectively, it is straightforward but somewhat tedious to show using the inequalities (9) that

$$\langle s_\alpha^3 s_\beta^3\rangle-\langle s_\alpha^3 s_\beta^3\rangle\leq 41\sum_{i\in\alpha}\sum_{j\in\beta}(\langle\sigma_i\sigma_j\rangle-\langle\sigma_i\rangle\langle\sigma_j\rangle). \tag{11}$$

Summing over the triangles (β) now yields

$$\chi_3\leq(3\times 6\times 41)\chi=738\chi, \tag{12}$$

where the numerical factor of 3 arises since there are 3 sites in triangle α, while the factor of 6 counts the number of triangles any other site is in, and hence the number of times that site will appear in the sum over triangles.

The result (12) is sufficient to prove that χ_3 cannot diverge at $K_2 = K_c$ any faster than χ. To prove that the critical exponents of χ_3 and χ are equal, however, requires a lower bound. This is easily achieved for temperatures above T_c (i.e. $K_2 < K_c$), but I have not been able to derive such a bound for $T < T_c$. Since above T_c, $\langle s_0^3 \rangle = M_3$ vanishes, the summand in (3) reduces in zero field to

$$\langle s_0^3 s_\alpha^3 \rangle = \langle \sigma_0^1 \sigma_0^2 \sigma_0^3 \sigma_\alpha^1 \sigma_\alpha^2 \sigma_\alpha^3 \rangle \geq \langle \sigma_0^1 \sigma_0^2 \rangle \langle \sigma_0^3 \sigma_\alpha^3 \rangle \langle \sigma_\alpha^1 \sigma_\alpha^2 \rangle = \Gamma_0^2 \langle \sigma_0^3 \sigma_\alpha^3 \rangle \geq 0, \quad (13)$$

where the intermediate inequality follows from the GKS inequality. Hence summing over triangles yields

$$\chi_3 \geq 2\Gamma_0^2 \chi. \quad (14)$$

Combining this result with (12) proves that the critical exponents of χ_3 and χ defined *above* T_c (i.e. in the limit $K_2 \to K_c -$) are identical.

In summary, we have proved for Ising models with ferromagnetic pair interactions that: (i) the triplet order parameter $M_3 = \langle \sigma_1 \sigma_2 \sigma_3 \rangle$ and the conventional singlet order parameter $M_0 = \langle \sigma_1 \rangle$ vanish at T_c with the same exponent in all dimensionalities; and (ii) the associated susceptibilities χ_3 and χ diverge with the same exponent as T_c is approached from above. We conjecture that this final result also holds as T_c is approached from below. The restriction to ferromagnetic pair interactions is necessitated by the use of the GHS inequality, which is restricted to this type of interaction (Griffiths *et al* 1970, Lebowitz 1974). Thus our proof of $\beta_3 = \beta$ does not apply to the face-centred and body-cubic lattices with pure quartet interactions also discussed by Wood and Griffiths (1976). Lebowitz (private communication) has, however, developed an alternative proof independent of the GHS inequality which is applicable to general Ising models, and thereby confirms the conjecture of Wood and Griffiths (1976) that $\beta_3 = \beta$ also for the case of quartet interactions.

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